

Generalized variational inclusion governed by generalized $\alpha\beta$ - $H((.,.), (.,.))$ -mixed accretive mapping in real q -uniformly smooth Banach spaces

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Abstract

In this paper, we investigate a new notion of accretive mappings called generalized $\alpha\beta$ - $H((.,.), (.,.))$ -mixed accretive mappings in Banach spaces. We extend the concept of proximal-point mappings associated with generalized m -accretive mappings to the generalized $\alpha\beta$ - $H((.,.), (.,.))$ -mixed accretive mappings and prove that the proximal-point mapping associated with generalized $\alpha\beta$ - $H((.,.), (.,.))$ -mixed accretive mapping is single-valued and Lipschitz continuous. Some examples are given to justify the definition of generalized $\alpha\beta$ - $H((.,.), (.,.))$ -mixed accretive mappings. Further, by using the proximal mapping technique, an iterative algorithm for solving a class of variational inclusions is constructed in real q -uniformly smooth Banach spaces. Under some suitable conditions, we prove the convergence of iterative sequence generated by the algorithm.

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1 Introduction

Variational inclusions, as the generalization of variational inequalities, have been widely studied in recent years. One of the most interesting and important problems in the theory of variational inclusions is the development of an efficient and implementable iterative algorithm. Variational inclusions include variational, quasi-variational, variational-like inequalities as special cases. For application of variational inclusions, see for example [1].

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Various kinds of iterative methods have been studied to find the approximate solutions for variational inclusions. Among these methods, The proximal-point mapping techniques are important to study the existence of solutions and to design iterative schemes for different kinds of variational inequalities and their generalizations, which are providing mathematical models to some problems arising in optimization and control, economics, and engineering sciences, has been widely used by many authors. For details, we refer to see [2]-[17], [20]-[23], [25, 26] and the references therein.

In order to study various variational inequalities and variational inclusions, Huang and Fang [7] were the first to introduce the generalized m -accretive mapping and give the definition of the proximal-point mapping in Banach spaces. Since then a number of researchers introduced several classes of generalized m -accretive mappings such as H -accretive, (H, η) -accretive, (A, η) -accretive, (P, η) -accretive mappings, see for examples [4]-[6], [11, 12, 16, 20, 21]. Sun et al. [22] introduced a new class of M -monotone mapping in Hilbert spaces. Recently, Zou and Huang [25, 26], Kazmi et al. [13, 14] introduced and studied a class of $H(.,.)$ -accretive mappings, Ahmad and Dilshad [2] introduced and studied a class of $H(.,.)$ - η -cocoercive and Husain and Gupta [8] introduced and studied a class of $H((.,.), (.,.))$ -mixed cocoercive mappings in Banach (Hilbert) spaces, an natural extension of M -monotone mapping and studied variational inclusions involving these mappings. In recent past, the methods based on different classes of proximal-point mappings have been developed to study the existence of solutions and to discuss convergence and stability analysis of iterative algorithms for various classes of variational inclusions, see for example [2, 3], [7]-[17], [20]-[23], [25].

Very recently, Luo and Huang [17] introduced and studied a class of B -monotone and Kazmi et al. [13] introduced and studied a class of generalized $H(.,.)$ -accretive mappings in Banach spaces, an extension of H -monotone mappings [4]. They showed some properties of the proximal-point mapping associated with B -monotone and generalized $H(.,.)$ -accretive mapping, and obtained some applications for solving variational inclusions in Banach spaces.

Motivated and inspired by the research works mentioned above, we consider a new class of generalized $\alpha\beta$ - $H((.,.), (.,.))$ -mixed accretive mappings for solving generalized set-valued variational inclusions in real q -uniformly smooth Banach spaces. We also define a proximal-point mapping associated with the generalized $\alpha\beta$ - $H((.,.), (.,.))$ -mixed accretive mapping and show that it is single-valued and Lipschitz continuous. By using the technique of proximal mapping, an iterative algorithm is constructed in Banach spaces. Under some suitable conditions, we prove the convergence of iterative sequence generated by the algorithm. The results presented in this paper can be viewed as an extension and generalization of some known results [2], [8]-[10], [13]-[15], [17, 23, 25, 26]. For illustration of Definitions 2.8, 3.1, and Theorem 4.6, Examples 3.2, 3.3 and 4.7 are given, respectively.

2 Preliminaries

Let X be a real Banach space equipped with the norm $\|\cdot\|$, and let X^* be the topological dual space of X . Let $\langle \cdot, \cdot \rangle$ be the dual pair between X and X^* , and let 2^X be the power set of X .

Definition 2.1 [24] For $q > 1$, a mapping $J_q : X \rightarrow 2^{X^*}$ is said to be a *generalized duality mapping*, if it is defined by

$$J_q(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in X.$$

In particular, J_2 is the *usual normalized duality mapping* on X . It is known that, in general

$$J_q(x) = \|x\|^{q-1} J_2(x) \quad \forall x (\neq 0) \in X.$$

If $X \equiv H$ a real Hilbert space, then J_2 becomes an *identity mapping* on H .

Definition 2.2 [24] A Banach space X is called *smooth* if, for every $x \in X$ with $\|x\| = 1$, there exists a unique $f \in X^*$ such that $\|f\| = f(x) = 1$.

The *modulus of smoothness* of X is a function $\rho_X : [0, \infty) \rightarrow [0, \infty)$, defined by

$$\rho_X(t) = \sup \left\{ \frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq t \right\}.$$

Definition 2.3 [24] A Banach space X is called

(i) *uniformly smooth* if

$$\lim_{t \rightarrow 0} \frac{\rho_X(t)}{t} = 0;$$

(ii) *q -uniformly smooth*, for $q > 1$, if there exists a constant $c > 0$ such that

$$\rho_X(t) \leq c t^q, \quad t \in [0, \infty).$$

Note that J_q is single-valued if X is uniformly smooth. Concerned with the characteristic inequalities in q -uniformly smooth Banach spaces, Xu [24] proved the following result.

Lemma 2.4 *Let X be a real uniformly smooth Banach space. Then X is q -uniformly smooth if and only if there exists a constant $c_q > 0$ such that, for all $x, y \in X$,*

$$\|x + y\|^q \leq \|x\|^q + q \langle y, J_q(x) \rangle + c_q \|y\|^q.$$

From Lemma 2 of Liu [18], it is easy to have the following lemma.

In the sequel, we recall important basic concepts and definitions, which will be used in this work.

Definition 2.5 A mapping $A : X \rightarrow X$ is said to be

(i) *accretive* if

$$\langle A(x) - A(y), J_q(x - y) \rangle \geq 0, \quad \forall x, y \in X;$$

(ii) *strictly accretive* if

$$\langle A(x) - A(y), J_q(x - y) \rangle > 0, \quad \forall x, y \in X;$$

and equality holds if and only if $x = y$;

(iii) ξ -*strongly accretive* if there exists a constant $\xi > 0$ such that

$$\langle A(x) - A(y), J_q(x - y) \rangle \geq \xi \|x - y\|^q, \quad \forall x, y \in X;$$

(iv) μ -*cocoercive* if there exists a constant $\mu > 0$ such that

$$\langle A(x) - A(y), J_q(x - y) \rangle \geq \mu \|A(x) - A(y)\|^q, \quad \forall x, y \in X;$$

(v) γ -*relaxed cocoercive* if there exists a constant $\gamma > 0$ such that

$$\langle A(x) - A(y), J_q(x - y) \rangle \geq -\gamma \|A(x) - A(y)\|^q, \quad \forall x, y \in X;$$

(vi) ζ -*Lipschitz continuous* if there exists a constant $\zeta > 0$ such that

$$\|A(x) - A(y)\| \leq \zeta \|x - y\|, \quad \forall x, y \in X;$$

(vii) α -*expansive* if there exists a constant $\alpha > 0$ such that

$$\|A(x) - A(y)\| \geq \alpha \|x - y\|, \quad \forall x, y \in X;$$

if $\alpha = 1$, then it is *expansive*.

Definition 2.6 [2, 25] Let $H : X \times X \rightarrow X$ and $A, B : X \rightarrow X$ be the single-valued mappings. Then

(i) $H(A, \cdot)$ is said to be α -*strongly accretive* with respect to A if there exists a constant $\alpha > 0$ such that

$$\langle H(Ax, u) - H(Ay, u), J_q(x - y) \rangle \geq \alpha \|x - y\|^q, \quad \forall x, y, u \in X;$$

(ii) $H(\cdot, B)$ is said to be β -*relaxed accretive* with respect to B if there exists a constant $\beta > 0$ such that

$$\langle H(u, Bx) - H(u, By), J_q(x - y) \rangle \geq -\beta \|x - y\|^q, \quad \forall x, y, u \in X;$$

(iii) $H(A, \cdot)$ is said to be μ -*cocoercive* with respect to A if there exists a constant $\mu > 0$ such that

$$\langle H(Ax, u) - H(Ay, u), J_q(x - y) \rangle \geq \mu \|Ax - Ay\|^q, \quad \forall x, y, u \in X;$$

(iv) $H(\cdot, B)$ is said to be γ -*relaxed cocoercive* with respect to B if there exists a constant $\gamma > 0$ such that

$$\langle H(u, Bx) - H(u, By), J_q(x - y) \rangle \geq -\gamma \|Bx - By\|^q, \quad \forall x, y, u \in X;$$

(v) $H(A, \cdot)$ is said to be τ_1 -Lipschitz continuous with respect to A if there exists a constant $\tau_1 > 0$ such that

$$\|H(Ax, \cdot) - H(Ay, \cdot)\| \leq \tau_1 \|x - y\|, \quad \forall x, y \in X;$$

(vi) $H(\cdot, B)$ is said to be τ_2 -Lipschitz continuous with respect to B if there exists a constant $\tau_2 > 0$ such that

$$\|H(\cdot, Bx) - H(\cdot, By)\| \leq \tau_2 \|x - y\|, \quad \forall x, y \in X.$$

Definition 2.7 [8] Let $H : (X \times X) \times (X \times X) \rightarrow X$, and $A, B, C, D : X \rightarrow X$ be the single-valued mappings. Then

(i) $H((A, \cdot), (C, \cdot))$ is said to be (μ_1, γ_1) -strongly mixed cocoercive with respect to (A, C) if there exist constants $\mu_1, \gamma_1 > 0$ such that

$$\langle H((Ax, u), (Cx, u)) - H((Ay, u), (Cy, u)), J_q(x - y) \rangle \geq \mu_1 \|Ax - Ay\|^q + \gamma_1 \|x - y\|^q, \quad \forall x, y, u \in X;$$

(ii) $H((\cdot, B), (\cdot, D))$ is said to be (μ_2, γ_2) -relaxed mixed cocoercive with respect to (B, D) if there exist constants $\mu_2, \gamma_2 > 0$ such that

$$\langle H((u, Bx), (u, Dx)) - H((u, By), (u, Dy)), J_q(x - y) \rangle \geq -\mu_2 \|Bx - By\|^q + \gamma_2 \|x - y\|^q, \quad \forall x, y, u \in X;$$

(iii) $H((A, B), (C, D))$ is said to be $\mu_1 \gamma_1 \mu_2 \gamma_2$ -symmetric mixed cocoercive with respect to (A, C) and (B, D) if $H((A, \cdot), (C, \cdot))$ is said to be (μ_1, γ_1) -strongly mixed cocoercive with respect to (A, C) and $H((\cdot, B), (\cdot, D))$ is said to be (μ_2, γ_2) -relaxed mixed cocoercive with respect to (B, D) ;

(iv) $H((A, B), (C, D))$ is said to be τ -mixed Lipschitz continuous with respect to A, B, C and D if there exists a constant $\tau > 0$ such that

$$\|H((Ax, Bx), (Cx, Dx)) - H((Ay, By), (Cy, Dy))\| \leq \tau \|x - y\|, \quad \forall x, y \in X.$$

Definition 2.8 [17] Let $T : X \multimap X$ and $M : X \multimap X$ be the set-valued mapping. Then

(i) T is said to be *accretive* if

$$\langle u - v, J_q(x - y) \rangle \geq 0 \quad \forall x, y \in X, u \in Tx, v \in Ty;$$

(ii) T is said to be *strictly accretive* if

$$\langle u - v, J_q(x - y) \rangle > 0 \quad \forall x, y \in X, u \in Tx, v \in Ty;$$

and equality holds if and only if $x = y$.

(iii) T is said to be μ' -strongly accretive if there exists a constant $\mu' > 0$ such that

$$\langle u - v, J_q(x - y) \rangle \geq \mu' \|x - y\|^q \quad \forall x, y \in X, u \in Tx, v \in Ty;$$

(iv) T is said to be γ' -relaxed accretive if there exists a constant $\gamma' > 0$ such that

$$\langle u - v, J_q(x - y) \rangle \geq -\gamma' \|x - y\|^q \quad \forall x, y \in X, u \in Tx, v \in Ty;$$

(v) $M(f, \cdot)$ is said to be α -strongly accretive with respect to f if there exists a constant $\alpha > 0$ such that

$$\langle u - v, J_q(x - y) \rangle \geq \alpha \|x - y\|^q \quad \forall x, y, w \in X, u \in M(f(x), w), v \in M(f(y), w);$$

(vi) $M(\cdot, g)$ is said to be β -relaxed accretive with respect to g if there exists a constant $\beta > 0$ such that

$$\langle u - v, J_q(x - y) \rangle \geq -\beta \|x - y\|^q \quad \forall x, y, w \in X, u \in M(w, g(x)), v \in M(w, g(y));$$

(vii) $M(\cdot, \cdot)$ is said to be $\alpha\beta$ -symmetric accretive with respect to f and g if $M(f, \cdot)$ is α -strongly accretive with respect to f and $M(\cdot, g)$ is β -relaxed accretive with respect to g with $\alpha \geq \beta$ and $\alpha = \beta$ if and only if $x = y$.

3 Generalized $\alpha\beta$ - $H((\cdot, \cdot), (\cdot, \cdot))$ -mixed accretive mappings

This section deals with a new concept and properties of generalized $\alpha\beta$ - $H((\cdot, \cdot), (\cdot, \cdot))$ -mixed accretive mappings, which provides a unifying framework for the existing cocoercive operators, accretive operators in Banach space and monotone operators in Hilbert space.

Definition 3.1 Let $H : (X \times X) \times (X \times X) \rightarrow X$, $f, g : X \rightarrow X$ and $A, B, C, D : X \rightarrow X$ be single-valued mappings. Let $H((A, B), (C, D))$ is $\mu_1\gamma_1\mu_2\gamma_2$ -symmetric mixed cocoercive with respect to (A, C) and (B, D) . Then the set-valued mapping $M : X \times X \rightarrow X$ is said to be generalized $\alpha\beta$ - $H((\cdot, \cdot), (\cdot, \cdot))$ -mixed accretive with respect to (A, C) , (B, D) and (f, g) if

- (i) M is $\alpha\beta$ -symmetric accretive with respect to f and g ;
- (ii) $(H((\cdot, \cdot), (\cdot, \cdot)) + \rho M(f, g))(X) = X$, for all $\rho > 0$.

The following example illustrate the Definitions (2.8) and (3.1).

Example 3.2 Let $q = 2$ and $X = \mathbb{R}^2$ with usual inner product defined by

$$\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_1 + x_2y_2.$$

Let $A, B, C, D : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$Ax = \begin{pmatrix} 4x_1 \\ 4x_2 \end{pmatrix}, Bx = \begin{pmatrix} -3x_1 \\ -3x_2 \end{pmatrix}, Cx = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}, Dx = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

Suppose that $H : (\mathbb{R}^2 \times \mathbb{R}^2) \times (\mathbb{R}^2 \times \mathbb{R}^2) \rightarrow \mathbb{R}^2$ is defined by

$$H((Ax, Bx), (Cx, Dx)) = Ax + Bx + Cx + Dx.$$

Then $H((A, B), (C, D))$ is symmetric mixed cocoercive with respect to (A, C) and (B, D) , and mixed Lipschitz continuous with respect to A, B, C and D .

Indeed, let for any $u \in \mathbb{R}^2$, we have

$$\begin{aligned}
& \langle H((Ax, u), (Cx, u)) - H((Ay, u), (Cy, u)), x - y \rangle \\
&= \langle Ax + Cx - Ay - Cy, x - y \rangle \\
&= \langle Ax - Ay, x - y \rangle + \langle Cx - Cy, x - y \rangle \\
&= \langle (4x_1 - 4y_1, 4x_2 - 4y_2), (x_1 - y_1, x_2 - y_2) \rangle \\
&\quad + \langle (2x_1 - 2y_1, 2x_2 - 2y_2), (x_1 - y_1, x_2 - y_2) \rangle \\
&= 4(x_1 - y_1)^2 + 4(x_2 - y_2)^2 + 2(x_1 - y_1)^2 + 2(x_2 - y_2)^2 \\
&= 4\|x - y\|^2 + 2\|x - y\|^2
\end{aligned}$$

and

$$\begin{aligned}
\|Ax - Ay\|^2 &= \langle Ax - Ay, Ax - Ay \rangle \\
&= \langle (4x_1 - 4x_2, 4y_1 - 4y_2), (4x_1 - 4x_2, 4y_1 - 4y_2) \rangle \\
&= 16(x_1 - y_1)^2 + 16(x_2 - y_2)^2,
\end{aligned}$$

that is,

$$\langle H((Ax, u), (Cx, u)) - H((Ay, u), (Cy, u)), x - y \rangle \geq \frac{1}{4} \|Ax - Ay\|^2 + 2\|x - y\|^2. \quad (3.1)$$

Hence, $H((A, B), (C, D))$ is $(\frac{1}{4}, 2)$ -strongly mixed cocoercive with respect to (A, C) .

$$\begin{aligned}
& \langle H((u, Bx), (u, Dx)) - H((u, By), (u, Dy)), x - y \rangle \\
&= \langle Bx + Dx - By - Dy, x - y \rangle \\
&= \langle Bx - By, x - y \rangle + \langle Dx - Dy, x - y \rangle \\
&= \langle (-3x_1 + 3y_1, -3x_2 + 3y_2), (x_1 - y_1, x_2 - y_2) \rangle \\
&\quad + \langle (x_1 - y_1, x_2 - y_2), (x_1 - y_1, x_2 - y_2) \rangle \\
&= -3(x_1 - y_1)^2 - 3(x_2 - y_2)^2 + (x_1 - y_1)^2 + (x_2 - y_2)^2 \\
&= -3\|x - y\|^2 + \|x - y\|^2
\end{aligned}$$

and

$$\begin{aligned}
\|Bx - By\|^2 &= \langle Bx - By, Bx - By \rangle \\
&= \langle (-3x_1 + 3y_1, -3x_2 + 3y_2), (-3x_1 + 3y_1, -3x_2 + 3y_2) \rangle \\
&= -9\{(x_1 - y_1)^2 + (x_2 - y_2)^2\},
\end{aligned}$$

that is,

$$\langle H((u, Bx), (u, Dx)) - H((u, By), (u, Dy)), x - y \rangle \geq -\frac{1}{3} \|Bx - By\|^2 + \|x - y\|^2. \quad (3.2)$$

Hence, $H((A, B), (C, D))$ is $(\frac{1}{3}, 1)$ -relaxed mixed cocoercive with respect to (B, D) .

From (3.1) and (3.2), $H((A, B), (C, D))$ is symmetric mixed cocoercive with respect to (A, C) and (B, D) .

$$\begin{aligned}
& \|H((Ax, Bx), (Cx, Dx)) - H((Ay, By), (Cy, Dy))\|^2 \\
&= \langle H((Ax, Bx), (Cx, Dx)) - H((Ay, By), (Cy, Dy)), \\
&\quad H((Ax, Bx), (Cx, Dx)) - H((Ay, By), (Cy, Dy)) \rangle \\
&= \langle (Ax + Bx + Cx + Dx) - (Ay + By + Cy + Dy), \\
&\quad (Ax + Bx + Cx + Dx) - (Ay + By + Cy + Dy) \rangle \\
&= \langle (4x_1, 4x_2) - (4y_1, 4y_2), (4x_1, 4x_2) - (4y_1, 4y_2) \rangle \\
&= 16\{(x_1 - y_1)^2 + (x_2 - y_2)^2\},
\end{aligned}$$

that is,

$$\|H((Ax, Bx), (Cx, Dx)) - H((Ay, By), (Cy, Dy))\| \leq 4\|x - y\|.$$

Hence, $H((A, B), (C, D))$ is mixed Lipschitz continuous with respect to A, B, C and D .

Now, we show that $M(f, g)$ is symmetric accretive with respect to f and g . Let $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$f(x) = \begin{pmatrix} 5x_1 - \frac{2}{3}x_2 \\ \frac{2}{3}x_1 + 5x_2 \end{pmatrix}, \quad g(x) = \begin{pmatrix} \frac{7}{4}x_1 + \frac{3}{4}x_2 \\ -\frac{3}{4}x_1 + \frac{7}{4}x_2 \end{pmatrix}, \quad \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

Suppose that $M : (\mathbb{R}^2 \times \mathbb{R}^2) \rightarrow \mathbb{R}^2$ is defined by

$$M(fx, gx) = fx - gx, \quad \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

Then $M(f, g)$ is symmetric accretive with respect to f and g , and M is generalized $\alpha\beta$ -mixed accretive with respect to $(A, B), (C, D)$ and (f, g) .

Let for any $w \in \mathbb{R}^2$, we have

$$\begin{aligned}
\langle M(fx, w) - M(fy, w), x - y \rangle &= \langle fx - w - fy + w, x - y \rangle \\
&= \langle fx - fy, x - y \rangle \\
&= \langle (5(x_1 - y_1) - \frac{2}{3}(x_2 - y_2)), (\frac{2}{3}(x_1 - y_1) + 5(x_2 - y_2)), \\
&\quad (x_1 - y_1, x_2 - y_2) \rangle \\
&= 5(x_1 - y_1)^2 + 5(x_2 - y_2)^2 \\
&= 5\|x - y\|^2,
\end{aligned}$$

that is,

$$\langle u - v, x - y \rangle \geq 5\|x - y\|^2, \quad \forall x, y \in X, u \in M(fx, w), v \in M(fy, w). \quad (3.3)$$

Hence, $M(f, g)$ is 5-strongly accretive with respect to f and

$$\begin{aligned}
\langle M(w, gx) - M(w, gy), x - y \rangle &= \langle w - gx - w + gy, x - y \rangle \\
&= -\langle gx - gy, x - y \rangle \\
&= -\langle (\frac{7}{4}(x_1 - y_1) + \frac{3}{4}(x_2 - y_2)), (-\frac{7}{4}(x_1 - y_1) + \frac{3}{4}(x_2 - y_2)), \\
&\quad (x_1 - y_1, x_2 - y_2) \rangle \\
&= -\{\frac{7}{4}(x_1 - y_1)^2 + \frac{7}{4}(x_2 - y_2)^2\} \\
&= -\frac{7}{4}\|x - y\|^2,
\end{aligned}$$

that is,

$$\langle u - v, x - y \rangle \geq -\frac{7}{4}\|x - y\|^2, \quad \forall x, y \in X, u \in M(w, gx), v \in M(w, gy). \quad (3.4)$$

Hence, $M(f, g)$ is $\frac{7}{4}$ -relaxed accretive with respect to g .

From (3.3) and (3.4), $M(f, g)$ is symmetric accretive with respect to f and g . Also for any $x \in \mathbb{R}^2$, we have

$$\begin{aligned}
[H((A, B), (C, D)) + \rho M(f, g)](x) &= [H((Ax, Bx), (Cx, Dx)) + \rho M(fx, gx)] \\
&= (Ax + Bx + Cx + Dx) + \rho(fx - gx) \\
&= (4x_1, 4x_2) + (-3x_1, -3x_2) + (2x_1, 2x_2) + (x_1, x_2) \\
&\quad + \rho\left(\left(5x_1 - \frac{2}{3}x_2, \frac{2}{3}x_1 + 5x_2\right) - \left(\frac{7}{4}x_1 - \frac{3}{4}x_2, \frac{3}{4}x_1 + \frac{7}{4}x_2\right)\right) \\
&= (4x_1, 4x_2) + \rho\left(\frac{13}{4}x_1 + \frac{1}{12}x_2, -\frac{1}{12}x_1 + \frac{13}{4}x_2\right) \\
&= \left(\left(4 + \frac{13}{4}\rho\right)x_1 + \frac{1}{12}\rho x_2, \left(4 - \frac{1}{12}\rho\right)x_1 + \frac{13}{4}\rho x_2\right),
\end{aligned}$$

it can be easily verify that the vector on right hand side generate the whole \mathbb{R}^2 . Therefore, we have

$$[H((A, B), (C, D)) + \rho M(f, g)]\mathbb{R}^2 = \mathbb{R}^2.$$

Hence, M is generalized $\alpha\beta$ - $H((\cdot, \cdot), (\cdot, \cdot))$ -mixed accretive with respect to (A, C) , (B, D) and (f, g) .

Example 3.3 Let $X = l^2$. Then inner product in l^2 is defined by

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i, \quad \forall x, y \in l^2.$$

Let $A, B, C, D, f, g : l^2 \rightarrow l^2$ be single-valued mappings are defined by

$$A(x) = -5x - 7e_n, \quad B(x) = 5x + 5e_n, \quad C(x) = -3x, \quad D(x) = 2x + 3e_n,$$

and $f(x) = 2x$, $g(x) = x - e_n$, $\forall x = (x_1, x_2, \dots, x_n, \dots) \in l^2$, where $e_n = (0, 0, \dots, 1, \dots) \in l^2$.

Let $H : l^2 \times l^2 \rightarrow l^2$ be a single-valued mapping defined by

$$H((Ax, Bx), (Cx, Dx)) = Ax + Bx + Cx + Dx, \quad \forall x \in l^2,$$

and let $M : l^2 \times l^2 \rightarrow l^2$ be a set-valued mapping defined by

$$M(f(x), g(x)) = f(x) - g(x).$$

Then

$$\begin{aligned} \langle M(f(x), w) - M(f(y), w), x - y \rangle &= \langle f(x) - w - g(y) + w, x - y \rangle \\ &= \langle 2x - 2y, x - y \rangle \\ &\geq 2 \|x - y\|^2 \end{aligned}$$

and

$$\begin{aligned} \langle M(w, g(x)) - M(w, g(y)), x - y \rangle &= \langle w - g(x) - w + g(y), x - y \rangle \\ &= -\langle x - e_n - y + e_n, x - y \rangle \\ &\geq -1 \|x - y\|^2. \end{aligned}$$

Then, for $\rho = 1$

$$\begin{aligned} \|[H((A, B), (C, D)) + \rho M(f, g)](x)\| &= \|Ax + Bx + Cx + Dx + f(x) - g(x)\| \\ &= \|-5x - 7e_n + 5x + 5e_n - 3x + 2x + 3e_n + 2x - x\| \\ &\geq \|e_n\|^2 = \langle e_n, e_n \rangle = \sum_{i=1}^{\infty} e_{n_i}^2 = 1 \end{aligned}$$

and so $0 \notin [H((A, B), (C, D)) + \rho M(f, g)](l^2)$. Thus, M is not a generalized $\alpha\beta$ - $H((A, B), (C, D))$ -mixed accretive mapping.

Remark 3.4 (i) If $H((A, B), (C, D)) = H(A, B)$, then generalized $\alpha\beta$ - $H((., .), (., .))$ -mixed accretive mapping reduces to generalized $H(., .)$ -accretive mapping considered in [15].

(ii) If $H((A, B), (C, D)) = B$, then generalized $\alpha\beta$ - $H((., .), (., .))$ -mixed accretive mapping reduces to generalized B -monotone mapping considered in [17].

(iii) If $H((A, B), (C, D)) = H(A, B)$, $M(., .) = M$ and M is η -cocoercive, then generalized $\alpha\beta$ - $H((., .), (., .))$ -mixed accretive mapping reduces to $H(., .)$ - η -cocoercive mapping considered in [2].

(iv) If $H((A, B), (C, D)) = H(A, B)$, $M(., .) = M$ and M is accretive, then generalized $\alpha\beta$ - $H((., .), (., .))$ -mixed accretive mapping reduces to $H(., .)$ -accretive mapping considered in [25].

(v) If $H((A, B), (C, D)) = H(\cdot, \cdot)$, $M(\cdot, \cdot) = M$ and M is accretive (monotone), then generalized $\alpha\beta$ - $H((\cdot, \cdot), (\cdot, \cdot))$ -mixed accretive mapping reduces to H -accretive mapping considered in [4, 5].

(vi) If X is Hilbert space, $M(f, g) = M$ and M is m -relaxed monotone, then generalized $\alpha\beta$ - $H((\cdot, \cdot), (\cdot, \cdot))$ -mixed accretive mapping reduces to $H((\cdot, \cdot), (\cdot, \cdot))$ -mixed cocoercive mapping considered in [8].

Since generalized $\alpha\beta$ - $H((\cdot, \cdot), (\cdot, \cdot))$ -mixed accretive mapping is a generalization of the maximal accretive mapping, it is sensible that there are similar properties between of them. The following result confirms this expectation.

Proposition 3.5 *Let the set-valued mapping $M : X \multimap X$ be a generalized $\alpha\beta$ - $H((\cdot, \cdot), (\cdot, \cdot))$ -mixed accretive mapping with respect to (A, C) , (B, D) and (f, g) . If A is α_1 -expansive, B is β_1 -Lipschitz continuous, and $\alpha > \beta$, $\mu_1 > \mu_2$, $\alpha_1 > \beta_1$ and $\gamma_1, \gamma_2 > 0$. Then the following inequality:*

$$\langle u - v, J_q(x - y) \rangle \geq 0,$$

holds for all $(v, y) \in \text{Graph}(M(f, g))$, implies $(u, x) \in M(f, g)$, where

$$\text{Graph}(M(f, g)) = \{(u, x) \in X \times X : (u, x) \in M(f(x), g(x))\}.$$

Proof. Suppose on the contrary that there exists $(u_0, x_0) \notin \text{Graph}(M(f, g))$ such that

$$\langle u_0 - v, J_q(x_0 - y) \rangle \geq 0, \forall (y, v) \in \text{Graph}(M(f, g)). \quad (3.5)$$

Since M is generalized $\alpha\beta$ - $H((\cdot, \cdot), (\cdot, \cdot))$ -mixed accretive with respect to (A, C) and (B, D) , we know that $(H((\cdot, \cdot), (\cdot, \cdot)) + \rho M(f, g))(X) = X$ holds for all $\rho > 0$ and so there exists $(u_1, x_1) \in \text{Graph}(M(f, g))$ such that

$$H((Ax_1, Bx_1), (Cx_1, Dx_1)) + \rho u_1 = H((Ax_0, Bx_0), (Cx_0, Dx_0)) + \rho u_0 \in X. \quad (3.6)$$

Now, $\rho u_0 - \rho u_1 = H((Ax_1, Bx_1), (Cx_1, Dx_1)) - H((Ax_0, Bx_0), (Cx_0, Dx_0)) \in X$.

$$\langle \rho u_0 - \rho u_1, J_q(x_0 - x_1) \rangle = -\langle H((Ax_0, Bx_0), (Cx_0, Dx_0)) - H((Ax_1, Bx_1), (Cx_1, Dx_1)), J_q(x_0 - x_1) \rangle.$$

Since M is $\alpha\beta$ -symmetric accretive with respect to f and g , we obtain

$$\begin{aligned} (\alpha - \beta) \|x_0 - x_1\|^q &\leq \rho \langle u_0 - u_1, J_q(x_0 - x_1) \rangle \\ &= -\langle H((Ax_0, Bx_0), (Cx_0, Dx_0)) - H((Ax_1, Bx_1), (Cx_1, Dx_1)), J_q(x_0 - x_1) \rangle \\ &= -\langle H((Ax_0, Bx_0), (Cx_0, Dx_0)) - H((Ax_1, Bx_0), (Cx_1, Dx_0)), J_q(x_0 - x_1) \rangle \\ &\quad -\langle H((Ax_1, Bx_0), (Cx_1, Dx_0)) - H((Ax_1, Bx_1), (Cx_1, Dx_1)), J_q(x_0 - x_1) \rangle. \end{aligned} \quad (3.7)$$

Since $H((A, B), (C, D))$ is $\mu_1\gamma_1\mu_2\gamma_2$ -symmetric mixed cocoercive with respect to (A, C) and (B, D) , thus (3.7) becomes

$$(\alpha - \beta) \|x_0 - x_1\|^q \leq -\mu_1 \|Ax_0 - Ax_1\|^q - \gamma_1 \|x_0 - x_1\|^q + \mu_2 \|Bx_0 - Bx_1\|^q - \gamma_2 \|x_0 - x_1\|^q. \quad (3.8)$$

Since A is α_1 -expansive and B is β_1 -Lipschitz continuous, thus (3.8) becomes

$$\begin{aligned} (\alpha - \beta) \|x_0 - x_1\|^q &\leq -\mu_1 \alpha_1^q \|x_0 - x_1\|^q - \gamma_1 \|x_0 - x_1\|^q + \mu_2 \beta_1^q \|x_0 - x_1\|^q - \gamma_2 \|x_0 - x_1\|^q \\ &= -[(\mu_1 \alpha_1^q - \mu_2 \beta_1^q) + (\gamma_1 + \gamma_2)] \|x_0 - x_1\|^q \\ 0 &\leq (\alpha - \beta) \|x_0 - x_1\|^q \leq -[(\mu_1 \alpha_1^q - \mu_2 \beta_1^q) + (\gamma_1 + \gamma_2)] \|x_0 - x_1\|^q \\ 0 &\leq -(r + m) \|x_0 - x_1\|^q \leq 0, \end{aligned}$$

where $r = (\mu_1 \alpha_1^q - \mu_2 \beta_1^q) + (\gamma_1 + \gamma_2)$ and $m = (\alpha - \beta)$,

which gives $x_0 = x_1$ since $\alpha > \beta$, $\mu_1 > \mu_2$, $\alpha_1 > \beta_1$ and $\gamma_1, \gamma_2 > 0$. By (3.6), we have $u_0 = u_1$, a contradiction. This complete the proof.

Theorem 3.6 Let the set-valued mapping $M : X \multimap X$ be a generalized $\alpha\beta$ - $H((\cdot, \cdot), (\cdot, \cdot))$ -mixed accretive mapping with respect to (A, C) , (B, D) and (f, g) . If A is α_1 -expansive, B is β_1 -Lipschitz continuous, and $\alpha > \beta$, $\mu_1 > \mu_2$, $\alpha_1 > \beta_1$ and $\gamma_1, \gamma_2 > 0$, then $(H((A, B), (C, D)) + \rho M(f, g))^{-1}$ is single-valued.

Proof. For any given $x \in X$, let $u, v \in (H((A, B), (C, D)) + \rho M(f, g))^{-1}(x)$. It follows that

$$\begin{cases} -H((Au, Bu), (Cu, Du)) + x \in \rho M(f, g)u, \\ -H((Av, Bv), (Cv, Dv)) + x \in \rho M(f, g)v. \end{cases}$$

Since M is $\alpha\beta$ -symmetric accretive with respect to f and g , we have

$$\begin{aligned} (\alpha - \beta) \|u - v\|^q &\leq \frac{1}{\rho} \langle -H((Au, Bu), (Cu, Du)) + x - (-H((Av, Bv), (Cv, Dv)) + x), J_q(u - v) \rangle \\ (\alpha - \beta) \|u - v\|^q &\leq \langle -H((Au, Bu), (Cu, Du)) + x - (-H((Av, Bv), (Cv, Dv)) + x), J_q(u - v) \rangle \\ &= \langle H((Au, Bu), (Cu, Du)) - H((Av, Bv), (Cv, Dv)), J_q(u - v) \rangle \\ &= \langle H((Au, Bu), (Cu, Du)) - H((Av, Bu), (Cv, Du)), J_q(u - v) \rangle \\ &\quad - \langle H((Av, Bu), (Cv, Du)) - H((Av, Bv), (Cv, Dv)), J_q(u - v) \rangle. \end{aligned} \tag{3.9}$$

Since $H((A, B), (C, D))$ is $\mu_1 \gamma_1 \mu_2 \gamma_2$ -symmetric mixed cocoercive with respect to (A, C) and (B, D) , thus (3.9) becomes

$$\rho(\alpha - \beta) \|u - v\|^q \leq -\mu_1 \|Au - Av\|^q - \gamma_1 \|u - v\|^q + \mu_2 \|Bu - Bv\|^q - \gamma_2 \|u - v\|^q. \tag{3.10}$$

Since A is α_1 -expansive and B is β_1 -Lipschitz continuous, thus (3.10) becomes

$$\begin{aligned} \rho(\alpha - \beta) \|u - v\|^q &\leq -\mu_1 \alpha_1^q \|u - v\|^q - \gamma_1 \|u - v\|^q + \mu_2 \beta_1^q \|u - v\|^q - \gamma_2 \|u - v\|^q \\ &= -[(\mu_1 \alpha_1^q - \mu_2 \beta_1^q) + (\gamma_1 + \gamma_2)] \|u - v\|^q \\ 0 &\leq (\alpha - \beta) \|u - v\|^q \leq -[(\mu_1 \alpha_1^q - \mu_2 \beta_1^q) + (\gamma_1 + \gamma_2)] \|u - v\|^q \\ 0 &\leq -(r + \rho m) \|u - v\|^q \leq 0, \end{aligned}$$

where $r = (\mu_1 \alpha_1^q - \mu_2 \beta_1^q) + (\gamma_1 + \gamma_2)$ and $m = (\alpha - \beta)$.

Since $\alpha > \beta$, $\mu_1 > \mu_2$, $\alpha_1 > \beta_1$ and $\gamma_1, \gamma_2 > 0$, it follows that $\|u - v\| \leq 0$. This implies that $u = v$ and so $(H((A, B), (C, D)) + \rho M(f, g))^{-1}$ is single-valued. \square

Definition 3.7 Let the set-valued mapping $M : X \multimap X$ be a generalized $\alpha\beta$ - $H((\cdot, \cdot), (\cdot, \cdot))$ -mixed accretive mapping with respect to (A, C) , (B, D) and (f, g) . If A is α_1 -expansive, B is β_1 -Lipschitz continuous, and $\alpha > \beta$, $\mu_1 > \mu_2$, $\alpha_1 > \beta_1$ and $\gamma_1, \gamma_2 > 0$, then the proximal-point mapping $R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))} : X \rightarrow X$ is defined by

$$R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(u) = (H((A, B), (C, D)) + \rho M(f, g))^{-1}(u), \quad \forall u \in X. \quad (3.11)$$

Remark 3.8 (i) If $H((A, B), (C, D)) = H(A, B)$, then the proximal-point mapping $R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}$ reduces to $R_{M(\cdot, \cdot), \rho}^{H(\cdot, \cdot)}$ considered in [15].

(ii) If $H((A, B), (C, D)) = B$, then the proximal-point mapping $R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}$ reduces to $R_{M(\cdot, \cdot), \rho}^B$ considered in [17].

(iii) If $H((A, B), (C, D)) = H(A, B)$, $M(f, g) = M$, and M is η -cocoercive, then the proximal-point mapping $R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}$ reduces to $R_{M, \rho}^{H(\cdot, \cdot) - \eta}$ considered in [2].

(iv) If $H((A, B), (C, D)) = H(A, B)$, $M(f, g) = M$, and M is accretive, then the proximal-point mapping $R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}$ reduces to $R_{M, \rho}^{H(\cdot, \cdot)}$ considered in [25].

(v) If $H((A, B), (C, D)) = H$, $M(f, g) = M$, and M is accretive (monotone), then the proximal-point mapping $R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}$ reduces to $R_{M, \rho}^H$ considered in [4, 5].

(vi) If X is Hilbert space, $M(f, g) = M$, and M is m -relaxed monotone, then the proximal-point mapping $R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}$ reduces to the resolvent operator $R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}$ considered in [8].

Now we prove that the proximal-point mapping defined by (3.11) is Lipschitz continuous.

Theorem 3.9 Let the set-valued mapping $M : X \multimap X$ be a generalized $\alpha\beta$ - $H((\cdot, \cdot), (\cdot, \cdot))$ -mixed accretive mapping with respect to (A, C) , (B, D) and (f, g) . If A is α_1 -expansive, B is β_1 -Lipschitz continuous, and $\alpha > \beta$, $\mu_1 > \mu_2$, $\alpha_1 > \beta_1$ and $\gamma_1, \gamma_2 > 0$, then the proximal-point mapping $R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))} : X \rightarrow X$ is $\frac{1}{r + \rho m}$ -Lipschitz continuous, that is,

$$\|R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(u) - R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(v)\| \leq \frac{1}{r + \rho m} \|u - v\|, \quad \forall u, v \in X.$$

Proof. Let $u, v \in X$ be any given points, It follows from (3.11) that

$$\begin{aligned} R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(u) &= (H((A, B), (C, D)) + \rho M(f, g))^{-1}(u), \\ R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(v) &= (H((A, B), (C, D)) + \rho M(f, g))^{-1}(v). \end{aligned}$$

$$\left\{ \begin{aligned} &\frac{1}{\rho} (u - H((A(R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(u)), B(R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(u))), \\ &\quad (C(R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(u)), D(R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(u)))) \in M(f(R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(u)), g(R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(u))) \\ &\frac{1}{\rho} (v - H((A(R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(v)), B(R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(v))), \\ &\quad (C(R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(v)), D(R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(v)))) \in M(f(R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(v)), g(R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(v))). \end{aligned} \right.$$

Since A is α_1 -expansive and B is β_1 -Lipschitz continuous, we have

$$\begin{aligned} \|u - v\| \|R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(u) - R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(v)\|^{q-1} &\geq [(\mu_1 \alpha_1^q - \mu_2 \beta_1^q) + (\gamma_1 + \gamma_2)] \|R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(u) - R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(v)\|^q \\ &\quad + \rho(\alpha - \beta) \|R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(u) - R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(v)\|^q \\ &\geq (r + \rho m) \|R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(u) - R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(v)\|^q, \end{aligned}$$

where $r = (\mu_1 \alpha_1^q - \mu_2 \beta_1^q) + (\gamma_1 + \gamma_2)$ and $m = (\alpha - \beta)$.

Hence,

$$\|u - v\| \|R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(u) - R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(v)\|^{q-1} \geq (r + \rho m) \|R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(u) - R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(v)\|^q,$$

that is

$$\|R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(u) - R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(v)\| \leq \frac{1}{r + \rho m} \|u - v\|, \quad \forall u, v \in X.$$

This completes the proof. \square

4 An application of generalized $\alpha\beta$ - $H((\cdot, \cdot), (\cdot, \cdot))$ -mixed accretive mappings for solving variational inclusions.

In this section, we shall show that under suitable assumptions, the generalized $\alpha\beta$ - $H((\cdot, \cdot), (\cdot, \cdot))$ -mixed accretive mapping can also play important roles for solving the generalized set-valued variational inclusion in Banach space.

Let $S, T : X \rightrightarrows CB(X)$ be the set-valued mappings, and let $f, g : X \rightarrow X, A, B, C, D : X \rightarrow X, F : X \times X \rightarrow X$ and $H : (X \times X) \times (X \times X) \rightarrow X$ be single-valued mappings. Suppose that $M : X \times X \rightrightarrows X$ is a set-valued mapping such that M be a generalized $\alpha\beta$ - $H((\cdot, \cdot), (\cdot, \cdot))$ -mixed accretive mapping with respect to $(A, C), (B, D)$ and (f, g) . We consider the following generalized set-valued variational inclusion: for given $\omega \in X$, find $u \in X, v \in S(u)$ and $w \in T(u)$ such that

$$\omega \in F(v, w) + M(f(u), g(u)). \quad (4.1)$$

If $S, T : X \rightarrow X$ be single-valued mappings and $M(\cdot, \cdot) = \lambda N(\cdot)$, where $\rho > 0$ is a constant, then the problem (4.1) reduces to the following problem: find $u \in X$ such that

$$\omega \in F(S(u), T(u)) + \lambda N(u). \quad (4.2)$$

If M is an (A, η) -accretive mapping, then the problem (4.2) was introduced and studied by Lan et al. [16].

If $\lambda = 1, a = 0$ and $F(S(u), T(u)) = T(u)$ for all $u \in X$, where $T : X \rightarrow X$ is a single-valued mapping, then the problem (4.2) reduces to the following problem: find $u \in X$ such that

$$0 \in T(u) + N(u). \quad (4.3)$$

If N is an $H(\cdot, \cdot)$ -accretive mapping, then the problem (4.3) was studied by Zou and Huang [25]; and N is a generalized m -accretive mapping, then the problem (4.3) was studied by Bi et al. [3].

If $X = H$ is a Hilbert space and N is an H -monotone mappings, then the problem (4.3) was introduced and studied by Fang and Huang [4] and includes many variational inequalities (inclusions) and complementarity problems as special cases. For example, see [20, 21].

Lemma 4.1 *Let $S, T : X \rightrightarrows CB(X)$ be the set-valued mappings, and let $f, g : X \rightarrow X$, $A, B, C, D : X \rightarrow X$, $F : X \times X \rightarrow X$ and $H : (X \times X) \times (X \times X) \rightarrow X$ be single-valued mappings. Suppose that $M : X \times X \rightrightarrows X$ is a set-valued mapping such that M be a generalized $\alpha\beta$ - $H((\cdot, \cdot), (\cdot, \cdot))$ -mixed accretive mapping with respect to (A, C) , (B, D) and (f, g) . Then $u \in X$, $v \in S(u)$ and $w \in T(u)$ is a solution of problem (4.1) if and only if $u \in X$, $v \in S(u)$ and $w \in T(u)$ satisfies the following relation:*

$$u = R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))} [H((Au, Bu), (Cu, Du)) - \rho F(v, w) + \rho \omega], \quad (4.4)$$

where $\rho > 0$ is a constant and $R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}$ is the proximal-point mapping defined by (3.11).

Proof. Observe that for $\rho > 0$,

$$\begin{aligned} & \omega \in F(w, v) + M(f(u), g(u)) \\ \Leftrightarrow & [H((Au, Bu), (Cu, Du)) - \rho F(v, w) + \rho \omega] \in H((Au, Bu), (Cu, Du)) + \rho M(f(u), g(u)) \\ \Leftrightarrow & [H((Au, Bu), (Cu, Du)) - \rho F(v, w) + \rho \omega] \in (H((A, B), (C, D)) + \rho M(f, g))u \\ \Leftrightarrow & u = (H((A, B), (C, D)) + \rho M(f, g))^{-1} [H((Au, Bu), (Cu, Du)) - \rho F(v, w) + \rho \omega] \\ \Leftrightarrow & u = R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))} [H((Au, Bu), (Cu, Du)) - \rho F(v, w) + \rho \omega]. \quad \square \end{aligned}$$

Remark 4.2 We can rewrite the equality (4.4) as:

$$z = H((Au, Bu), (Cu, Du)) - \rho F(v, w) + \rho \omega, \quad u = R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(z).$$

where $\omega \in X$ is any given element and $\rho > 0$ is a constant. By Nadler [19], we know that this fixed point formulation enables us to suggest the following iterative algorithm.

Algorithm 4.3 For any given $z_0 \in X$, we can choose $u_0 \in X$ such that sequences $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ satisfy

$$\left\{ \begin{array}{l} u_n = R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(z_n), \\ v_n \in S(u_n), \quad \|v_n - v_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right) \mathcal{D}(S(u_n), S(u_{n+1})), \\ w_n \in T(u_n), \quad \|w_n - w_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right) \mathcal{D}(T(u_n), T(u_{n+1})), \\ z_{n+1} = H((Au_n, Bu_n), (Cu_n, Du_n)) - \rho F(v_n, w_n) + \rho \omega + e_n, \\ \sum_{j=1}^{\infty} \|e_j - e_{j-1}\| \omega^{-j} < \infty, \quad \forall \omega \in (0, 1), \quad \lim_{n \rightarrow \infty} e_n = 0, \end{array} \right.$$

where $\rho > 0$ is a constant, $\omega \in X$ is any given element and $e_n \subset X$ is an error to take into account a possible inexact computation of the proximal-point mapping point for all $n \geq 0$, and $\mathcal{D}(\cdot, \cdot)$ is the Hausdorff metric on $CB(X)$.

We need the following definitions which will be used to state and prove the main result.

Definition 4.4 A set-valued mapping $G : X \multimap CB(X)$ is said to be \mathcal{D} -Lipschitz continuous if there exists a constant $l > 0$ such that

$$\mathcal{D}(Gx, Gy) \leq l \|x - y\|, \quad \forall x, y \in X.$$

Definition 4.5 Let $S, T : X \multimap X$ be the set-valued mappings, $A, B, C, D : X \rightarrow X$, $F : X \times X \rightarrow X$ and $H : (X \times X) \times (X \times X) \rightarrow X$ be single-valued mappings. Then

(i) F is said to be σ -strongly accretive with respect to S and $H((A, B), (C, D))$ in the first argument if there exists a constant $\sigma > 0$ such that

$$\begin{aligned} \langle F(v_1, \cdot) - F(v_2, \cdot), J_q(H((Au, Bu), (Cu, Du)) - H((Av, Bv), (Cv, Dv))) \rangle \\ \geq \sigma \|H((Au, Bu), (Cu, Du)) - H((Av, Bv), (Cv, Dv))\|^q, \\ \forall u, v \in X \text{ and } v_1 \in S(u), v_2 \in S(v); \end{aligned}$$

(ii) F is said to be δ -strongly accretive with respect to T and $H((A, B), (C, D))$ in the second argument if there exists a constant $\delta > 0$ such that

$$\begin{aligned} \langle F(\cdot, w_1) - F(\cdot, w_2), J_q(H((Au, Bu), (Cu, Du)) - H((Av, Bv), (Cv, Dv))) \rangle \\ \geq \delta \|H((Au, Bu), (Cu, Du)) - H((Av, Bv), (Cv, Dv))\|^q, \\ \forall u, v \in X \text{ and } w_1 \in T(u), w_2 \in T(v); \end{aligned}$$

(iii) F is said to be ϵ_1 -Lipschitz continuous in the first argument if there exists a constant $\epsilon_1 > 0$ such that

$$\|F(u, v') - F(v, v')\| \leq \epsilon_1 \|u - v\|, \quad \forall u, v, v' \in X;$$

(iv) F is said to be ϵ_2 -Lipschitz continuous in the second argument if there exists a constant $\epsilon_2 > 0$ such that

$$\|F(v', u) - F(v', v)\| \leq \epsilon_2 \|u - v\|, \quad \forall u, v, v' \in X.$$

Next, we find the convergence of iterative algorithm for generalized set-valued variational inclusion (4.1).

Theorem 4.6 Let $S, T : X \multimap CB(X)$ be the set-valued mappings, and let $f, g : X \rightarrow X$, $A, B, C, D : X \rightarrow X$, $F : X \times X \rightarrow X$ and $H : (X \times X) \times (X \times X) \rightarrow X$ be single-valued mappings. Suppose that $M : X \times X \multimap X$ is a set-valued mapping such that M be a generalized $\alpha\beta$ - $H((\cdot, \cdot), (\cdot, \cdot))$ -mixed accretive mapping with respect to (A, C) , (B, D) and (f, g) . Assume that

- (i) S and T are \mathcal{D} -Lipschitz continuous with constants l_1 and l_2 , respectively;
- (ii) A is α_1 -expansive and B is β_1 -Lipschitz continuous;
- (iii) $H((A, B), (C, D))$ is τ -mixed Lipschitz continuous with respect to A, B, C and D ;

(iv) F is σ -strongly accretive with respect to S and $H((A, B), (C, D))$ in the first argument and δ -strongly accretive with respect to T and $H((A, B), (C, D))$ in the second argument;

(v) F is ϵ_1 -Lipschitz continuous in the first argument and ϵ_2 -Lipschitz continuous in the second argument;

$$(vi) 0 < \sqrt[q]{\tau^q + c_q \rho^q (\epsilon_1 l_1 + \epsilon_2 l_2)^q} - \rho q (\sigma + \delta) \tau^q < r + \rho m; \quad (4.5)$$

where $r = (\mu_1 \alpha_1^q - \mu_2 \beta_1^q) + (\gamma_1 + \gamma_2)$ and $m = \alpha - \beta$, and $\alpha > \beta$, $\mu_1 > \mu_2$, $\alpha_1 > \beta_1$ and $\gamma_1, \gamma_2, \rho > 0$.

Then generalized set-valued variational inclusion problem (4.1) has a solution (u, v, w) , where $u \in X$, $v \in S(u)$ and $w \in T(u)$, and the iterative sequences $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$, generated by Algorithms 4.3 converges strongly to u , v and w , respectively.

Proof. Since S and T are \mathcal{D} -Lipschitz continuous with constants l_1 and l_2 , respectively, it follows from Algorithms 4.3 such that

$$\|v_{n+1} - v_n\| \leq \left(1 + \frac{1}{n+1}\right) \mathcal{D}(S(u_{n+1}), S(u_n)) \leq \left(1 + \frac{1}{n+1}\right) l_1 \|u_{n+1} - u_n\|, \quad (4.6)$$

$$\|w_{n+1} - w_n\| \leq \left(1 + \frac{1}{n+1}\right) \mathcal{D}(T(u_{n+1}), T(u_n)) \leq \left(1 + \frac{1}{n+1}\right) l_2 \|u_{n+1} - u_n\|, \quad (4.7)$$

for $n = 0, 1, 2, \dots$

It follows from (4.4) and Theorem 3.8 that

$$\|u_{n+1} - u_n\| \leq \|R_{\rho, M(\cdot, \cdot)}^{H(\cdot, \cdot), (\cdot, \cdot)}(z_{n+1}) - R_{\rho, M(\cdot, \cdot)}^{H(\cdot, \cdot), (\cdot, \cdot)}(z_n)\| = \frac{1}{r + \rho m} \|z_{n+1} - z_n\|. \quad (4.8)$$

Now, we estimate $\|z_{n+1} - z_n\|$ by using Algorithms 4.3, we have

$$\begin{aligned} \|z_{n+1} - z_n\| &= \| [H((Au_n, Bu_n), (Cu_n, u_n)) - \rho F(v_n, w_n) + \rho \omega + e_n] \\ &\quad - [H((Au_{n-1}, Bu_{n-1}), (Cu_{n-1}, u_{n-1})) - \rho F(v_{n-1}, w_{n-1}) + \rho \omega + e_{n-1}] \| \\ &\leq \|H((Au_n, Bu_n), (Cu_n, u_n)) - H((Au_{n-1}, Bu_{n-1}), (Cu_{n-1}, u_{n-1}))\| \\ &\quad + \|\rho F(v_n, w_n) - \rho F(v_{n-1}, w_{n-1})\| + \|e_n - e_{n-1}\|. \end{aligned} \quad (4.9)$$

By Lemma 2.4, we have

$$\begin{aligned} &\|H((Au_n, Bu_n), (Cu_n, Du_n)) - H((Au_{n-1}, Bu_{n-1}), (Cu_{n-1}, Du_{n-1})) - \rho(F(v_n, w_n) - F(v_{n-1}, w_{n-1}))\|^q \\ &\leq \|H((Au_n, Bu_n), (Cu_n, Du_n)) - H((Au_{n-1}, Bu_{n-1}), (Cu_{n-1}, Du_{n-1}))\|^q + c_q \rho^q \|F(v_n, w_n) - F(v_{n-1}, w_{n-1})\|^q \\ &\quad - \rho q \langle F(v_n, w_n) - F(v_{n-1}, w_{n-1}), J_q(H((Au_n, Bu_n), (Cu_n, Du_n)) - H((Au_{n-1}, Bu_{n-1}), (Cu_{n-1}, Du_{n-1}))) \rangle. \end{aligned} \quad (4.10)$$

From (iii), we get

$$\|H((Au_n, Bu_n), (Cu_n, Du_n)) - H((Au_{n-1}, Bu_{n-1}), (Cu_{n-1}, Du_{n-1}))\| \leq \tau \|u_n - u_{n-1}\|. \quad (4.11)$$

Using Algorithm 4.3, and conditions (i) and (v), we get

$$\begin{aligned}
\|F(v_n, w_n) - F(v_{n-1}, w_{n-1})\| &\leq \|F(v_n, w_n) - F(v_{n-1}, w_n)\| + \|F(v_{n-1}, w_n) - F(v_{n-1}, w_{n-1})\| \\
&\leq \epsilon_1 \|v_n - v_{n-1}\| + \epsilon_2 \|w_n - w_{n-1}\| \\
&\leq \epsilon_1 \left(1 + \frac{1}{n}\right) \mathcal{D}(S(u_n), S(u_{n-1})) + \epsilon_2 \left(1 + \frac{1}{n}\right) \mathcal{D}(T(u_n), T(u_{n-1})) \\
&\leq \left(\epsilon_1 l_1 \left(1 + \frac{1}{n}\right) + \epsilon_2 l_2 \left(1 + \frac{1}{n}\right) \right) \|u_n - u_{n-1}\|. \tag{4.12}
\end{aligned}$$

Using conditions (iv), we get

$$\begin{aligned}
&\langle F(v_n, w_n) - F(v_{n-1}, w_{n-1}), J_q(H((Au_n, Bu_n), (Cu_n, Du_n)) - H((Au_{n-1}, Bu_{n-1}), (Cu_{n-1}, Du_{n-1}))) \rangle \\
&\leq \langle F(v_n, w_n) - F(v_{n-1}, w_n), J_q(H((Au_n, Bu_n), (Cu_n, Du_n)) - H((Au_{n-1}, Bu_{n-1}), (Cu_{n-1}, Du_{n-1}))) \rangle \\
&\quad + \langle F(v_{n-1}, w_n) - F(v_{n-1}, w_{n-1}), J_q(H((Au_n, Bu_n), (Cu_n, Du_n)) - H((Au_{n-1}, Bu_{n-1}), (Cu_{n-1}, Du_{n-1}))) \rangle \\
&\leq (\sigma + \delta) \|H((Au_n, Bu_n), (Cu_n, Du_n)) - H((Au_{n-1}, Bu_{n-1}), (Cu_{n-1}, Du_{n-1}))\|^q \\
&\leq (\sigma + \delta) \tau^q \|u_n - u_{n-1}\|^q. \tag{4.13}
\end{aligned}$$

From (4.10)-(4.13), we have

$$\begin{aligned}
&\|H((Au_n, Bu_n), (Cu_n, Du_n)) - H((Au_{n-1}, Bu_{n-1}), (Cu_{n-1}, Du_{n-1})) - \rho(F(v_n, w_n) - F(v_{n-1}, w_{n-1}))\| \\
&\leq \sqrt[q]{\tau^q + c_q \rho^q \left(\epsilon_1 l_1 \left(1 + \frac{1}{n}\right) + \epsilon_2 l_2 \left(1 + \frac{1}{n}\right) \right)^q - \rho q (\sigma + \delta) \tau^q \|u_n - u_{n-1}\|}. \tag{4.14}
\end{aligned}$$

Combining (4.8), (4.9) and (4.14), we have

$$\begin{aligned}
\|u_{n+1} - u_n\| &\leq \|R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(z_{n+1}) - R_{\rho, M(\cdot, \cdot)}^{H((\cdot, \cdot), (\cdot, \cdot))}(z_n)\| \\
&\leq \theta_n \|u_n - u_{n-1}\| + \frac{1}{r + \rho m} \|e_n - e_{n-1}\|, \tag{4.15}
\end{aligned}$$

where

$$\theta_n = \frac{1}{r + \rho m} \sqrt[q]{\tau^q + c_q \rho^q \left(\epsilon_1 l_1 \left(1 + \frac{1}{n}\right) + \epsilon_2 l_2 \left(1 + \frac{1}{n}\right) \right)^q - \rho q (\sigma + \delta) \tau^q}. \tag{4.16}$$

Let

$$\theta = \frac{1}{r + \rho m} \sqrt[q]{\tau^q + c_q \rho^q (\epsilon_1 l_1 + \epsilon_2 l_2)^q - \rho q (\sigma + \delta) \tau^q}. \tag{4.17}$$

Then we know that $\theta_n \rightarrow \theta$ as $n \rightarrow \infty$. By (4.5), we know that $0 < \theta < 1$ and hence there exist $n_0 > 0$ and $\theta_0 \in (0, 1)$ such that $\theta_n \leq \theta_0$ for all $n \geq n_0$. Therefore, by (4.15), we have

$$\|u_{n+1} - u_n\| \leq \theta_0 \|u_n - u_{n-1}\| + \frac{1}{r + \rho m} \|e_n - e_{n-1}\| \quad \forall n \geq n_0. \tag{4.18}$$

(4.18) implies that

$$\|u_{n+1} - u_n\| \leq \theta_0^{n-n_0} \|u_{n_0+1} - u_{n_0}\| + \frac{1}{r + \rho m} \sum_{j=1}^{n-n_0} \theta_0^{j-1} t_{n-(n-1)}, \quad (4.19)$$

where $t_n = \|e_n - e_{n-1}\|$ for all $n \geq n_0$. Hence, for any $m \geq n > n_0$, we have

$$\begin{aligned} \|u_m - u_n\| &\leq \sum_{p=n}^{m-1} \|u_{p+1} - u_p\| \\ &\leq \sum_{p=n}^{m-1} \theta_0^{p-n_0} \|u_{n_0+1} - u_{n_0}\| + \frac{1}{r + \rho m} \sum_{p=n}^{m-1} \theta_0^p \sum_{j=1}^{p-n_0} \left[\frac{t_{p-(j-1)}}{\theta_0^{p-(j-1)}} \right]. \end{aligned} \quad (4.20)$$

Since $\sum_{j=1}^{\infty} \|e_j - e_{j-1}\| \omega^{-j} < \infty$, $\forall \omega \in (0, 1)$ and $0 < \theta_0 < 1$, it follows that $\|u_m - u_n\| \rightarrow 0$ as $n \rightarrow \infty$, and so $\{u_n\}$ is a Cauchy sequence in X . From (4.6) and (4.7), it follows that $\{v_n\}$ and $\{w_n\}$ are also Cauchy sequences in X . Thus, there exist u, v and w such that $u_n \rightarrow u$, $v_n \rightarrow v$ and $w_n \rightarrow w$ as $n \rightarrow \infty$. In the sequel, we will prove that $v \in S(u)$. In fact, since $v_n \in S(u_n)$, we have

$$\begin{aligned} d(v, S(u)) &\leq \|v - v_n\| + d(v_n, S(u)) \\ &\leq \|v - v_n\| + \mathcal{D}(S(u_n), S(u)) \\ &\leq \|v - v_n\| + \rho \|u_n - u\| \rightarrow 0, \text{ as } n \rightarrow \infty, \end{aligned}$$

which implies that $d(v, S(u)) = 0$. Since $S(u) \in CB(X)$, it follows that $v \in S(u)$. Similarly, it is easy to see that $w \in T(u)$.

By the continuity of $R_{\rho, M(\cdot, \cdot)}^{H(\cdot, \cdot), (\cdot, \cdot)}$, A, B, C, D, S, T and F and Algorithms 4.3, we know that u, v and w satisfy

$$u = R_{\rho, M(\cdot, \cdot)}^{H(\cdot, \cdot), (\cdot, \cdot)} [H((Au, Bu), (Cu, Du)) - \rho F(u, z) + \rho \omega].$$

By Lemma 4.1, (u, v, w) is a solution of the problem (4.1). This completes the proof \square

The following example shows that assumptions (i) to (vi) of Theorem 4.6 are satisfied for variational inclusion problem (4.1).

Example 4.7 Let $q = 2$ and $X = \mathbb{R}^2$ with usual inner product.

(i) Let $S, T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are identity mappings, then R, S are n -Lipschitz continuous for $n = 1, 2$.

Let $A, B, C, D : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$Ax = \begin{pmatrix} \frac{1}{10}x_1 \\ \frac{1}{10}x_2 \end{pmatrix}, \quad Bx = \begin{pmatrix} -\frac{1}{5}x_1 \\ -\frac{1}{5}x_2 \end{pmatrix}, \quad Cx = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}, \quad Dx = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

Suppose that $H : (\mathbb{R}^2 \times \mathbb{R}^2) \times (\mathbb{R}^2 \times \mathbb{R}^2) \rightarrow \mathbb{R}^2$ is defined by

$$H((Ax, By), (Cx, Dy)) = Ax + Bx + Cx + Dx, \quad \forall x \in \mathbb{R}^2.$$

Then, it is easy to check that

(ii) $H((\cdot, \cdot), (\cdot, \cdot))$ is $(10, 2)$ -strongly mixed cocoercive with respect to (A, C) and $(5, 1)$ -relaxed mixed cocoercive with respect to (B, D) , and A is $\frac{1}{n}$ -expansive for $n = 10, 11$ and B is $\frac{1}{n}$ -Lipschitz continuous for $n = 4, 5$.

(iii) $H((A, B), (C, D))$ is $\frac{29}{n}$ -mixed Lipschitz continuous with respect to A, B, C and D for $n = 9, 10$.

Let $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$f(x) = \begin{pmatrix} \frac{1}{2}x_1 - \frac{4}{3}x_2 \\ \frac{4}{3}x_1 + \frac{1}{2}x_2 \end{pmatrix}, \quad g(x) = \begin{pmatrix} \frac{1}{4}x_1 - \frac{3}{4}x_2 \\ \frac{3}{4}x_1 + \frac{1}{4}x_2 \end{pmatrix}, \quad \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

Suppose that $M : (\mathbb{R}^2 \times \mathbb{R}^2) \rightarrow \mathbb{R}^2$ is defined by

$$M(fx, gx) = fx - gx, \quad \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

Then, it is easy to check that $M(f, g)$ is $\frac{1}{n}$ -strongly accretive with respect to f for $n = 2, 3$ and $\frac{1}{n}$ -relaxed accretive with respect to g for $n = 3, 4$. Moreover, for $\rho = 1$, M is generalized $\alpha\beta$ - $H((\cdot, \cdot), (\cdot, \cdot))$ -mixed accretive with respect to (A, C) , (B, D) and (f, g) .

Let $F : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are defined by

$$F(x, y) = \frac{x}{4} + \frac{y}{5}, \quad \forall x, y \in \mathbb{R}^2.$$

Then, it is easy to check that

(iv) F is $\frac{29}{n}$ -strongly accretive with respect to S and $H((A, B), (C, D))$ in the first argument for $n = 30, 40$ and $\frac{29}{n}$ -strongly accretive with respect to T and $H((A, B), (C, D))$ in the second argument for $n = 40, 50$;

(v) F is $\frac{1}{n}$ -Lipschitz continuous in the first argument for $n = 3, 4$ and $\frac{1}{n}$ -Lipschitz continuous in the second argument for $n = 4, 5$.

Therefore, for the constants

$$\begin{aligned} l_1 = l_2 = 1, \quad \mu_1 = 10, \quad \gamma_1 = 2, \quad \mu_2 = 5, \quad \gamma_2 = 1, \quad \alpha_1 = 0.1, \quad \beta_1 = 0.2, \\ \alpha = 0.5, \quad \beta = 0.25, \quad \sigma = 0.725, \quad \delta = 0.580, \quad \epsilon_1 = 0.25, \quad \epsilon_2 = 0.2, \quad \tau = 2.9, \\ q = 2, \quad r = 2.9, \quad m = 0.25. \end{aligned}$$

obtained in (i) to (viii) above, all the conditions of the Theorem 4.7 is satisfied for the generalized mixed variational inclusion problem (4.1) for $\rho = 0.35$ and $c_q = 1$.

Remark 4.8 If the set-valued mapping $M(f, g)$ is η -accretive and $\eta(x, y)$ is Lipschitz continuous, then M becomes a new generalized $\alpha\beta$ - $H((\cdot, \cdot), (\cdot, \cdot))$ - η -mixed accretive mapping, see e.g. [23]. We leave the proofs to readers who are interested in this area.

Appendix: Verification (Calculations) of Example 4.7

Example 4.7 Let $q = 2$ and $X = \mathbb{R}^2$ with usual inner product.

(i) Let $S, T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are identity mappings, then R, S are n -Lipschitz continuous for $n = 1, 2$.

(ii) Let $A, B, C, D : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$Ax = \begin{pmatrix} \frac{1}{10}x_1 \\ \frac{1}{10}x_2 \end{pmatrix}, Bx = \begin{pmatrix} -\frac{1}{5}x_1 \\ -\frac{1}{5}x_2 \end{pmatrix}, Cx = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}, Dx = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

Suppose that $H : (\mathbb{R}^2 \times \mathbb{R}^2) \times (\mathbb{R}^2 \times \mathbb{R}^2) \rightarrow \mathbb{R}^2$ is defined by

$$H((Ax, By), (Cx, Dy)) = Ax + Bx + Cx + Dx, \quad \forall x \in \mathbb{R}^2.$$

Then

$$\begin{aligned} & \langle H((Ax, u), (Cx, u)) - H((Ay, u), (Cy, u)), x - y \rangle \\ &= \langle (Ax + u + Cx + u) - (Ay + u + Cy + u), x - y \rangle \\ &= \langle Ax - Ay, x - y \rangle + \langle Cx - Cy, x - y \rangle \\ &= \langle (\frac{1}{10}x_1 - \frac{1}{10}y_1, \frac{1}{10}x_2 - \frac{1}{10}y_2), (x_1 - y_1, x_2 - y_2) \rangle \\ &\quad + \langle (2x_1 - 2y_1, 2x_2 - 2y_2), (x_1 - y_1, x_2 - y_2) \rangle \\ &= \frac{1}{10}\|x - y\|^2 + 2\|x - y\|^2 \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} \|Ax - Ay\|^2 &= \langle Ax - Ay, Ax - Ay \rangle \\ &= \langle (\frac{1}{10}x_1 - \frac{1}{10}y_1, \frac{1}{10}x_2 - \frac{1}{10}y_2), (\frac{1}{10}x_1 - \frac{1}{10}y_1, \frac{1}{10}x_2 - \frac{1}{10}y_2) \rangle \\ &= \frac{1}{100}\{(x_1 - y_1)^2 + (x_2 - y_2)^2\} \\ &= \frac{1}{100}\|x - y\|^2. \end{aligned} \tag{4.22}$$

From (4.22) and (4.23), we have

$$\langle H((Ax, u), (Cx, u)) - H((Ay, u), (Cy, u)), x - y \rangle \geq 10\|Ax - Ay\|^2 + 2\|x - y\|^2. \tag{4.23}$$

Let

$$\begin{aligned} & \langle H((u, Bx), (u, Dx)) - H((u, By), (u, Dy)), x - y \rangle \\ &= \langle (u + Bx + u + Dx) - (u + By + u + Dy), x - y \rangle \\ &= \langle Bx - By, x - y \rangle + \langle Dx - Dy, x - y \rangle \\ &= \langle (-\frac{1}{5}x_1 - \frac{1}{5}y_1, -\frac{1}{5}x_2 - \frac{1}{5}y_2), (x_1 - y_1, x_2 - y_2) \rangle \\ &\quad + \langle (x_1 - y_1, x_2 - y_2), (x_1 - y_1, x_2 - y_2) \rangle \end{aligned}$$

$$= -\frac{1}{5}\|x - y\|^2 + \|x - y\|^2 \quad (4.24)$$

and

$$\begin{aligned} \|Bx - By\|^2 &= \langle Ax - Ay, Ax - Ay \rangle \\ &= \langle (-\frac{1}{10}x_1 - \frac{1}{5}y_1), -(\frac{1}{5}x_2 - \frac{1}{5}y_2) \rangle, (-\frac{1}{5}x_1 - \frac{1}{5}y_1), -(\frac{1}{5}x_2 - \frac{1}{5}y_2) \rangle \\ &= -\frac{1}{25}\{(x_1 - y_1)^2 + (x_2 - y_2)^2\} \\ &= -\frac{1}{25}\|x - y\|^2. \end{aligned} \quad (4.25)$$

From (4.24) and (4.25), we have

$$\langle H((Ax, u), (Cx, u)) - H((Ay, u), (Cy, u)), x - y \rangle \geq -5\|Ax - Ay\|^2 + \|x - y\|^2. \quad (4.26)$$

From (4.22)-(4.25), $H((\cdot, \cdot), (\cdot, \cdot))$ is $(10, 2)$ -strongly mixed cocoercive with respect to (A, C) and $(5, 1)$ -relaxed mixed cocoercive with respect to (B, D) , respectively.

$$\|Ax - Ay\| \geq \frac{1}{n}\|x - y\|^2, \text{ for } n = 10, 11, \quad (4.27)$$

$$\|Bx - By\| \leq -\frac{1}{n}\|x - y\|^2, \text{ for } n = 5, 6. \quad (4.28)$$

From (4.27) and (4.28), A is $\frac{1}{n}$ -expansive for $n = 10, 11$ and B is $\frac{1}{n}$ -Lipschitz continuous for $n = 4, 5$, respectively.

(iii) Let

$$\begin{aligned} &\|H((Ax, Bx), (Cx, Dx)) - H((Ay, By), (Cy, Dy))\|^2 \\ &= \langle H((Ax, Bx), (Cx, Dx)) - H((Ay, By), (Cy, Dy)), \\ &\quad H((Ax, Bx), (Cx, Dx)) - H((Ay, By), (Cy, Dy)) \rangle \\ &= \langle (Ax + Bx + Cx + Dx) - (Ay + By + Cy + Dy), \\ &\quad (Ax + Bx + Cx + Dx) - (Ay + By + Cy + Dy) \rangle \\ &= \langle (\frac{29}{10}x_1 - \frac{29}{10}y_1, \frac{29}{10}x_2 - \frac{29}{10}y_2), (\frac{29}{10}x_1 - \frac{29}{10}y_1, \frac{29}{10}x_2 - \frac{29}{10}y_2) \rangle \\ &= \frac{29^2}{10^2}\{(x_1 - y_1)^2 + (x_2 - y_2)^2\}, \end{aligned}$$

that is,

$$\|H((Ax, Bx), (Cx, Dx)) - H((Ay, By), (Cy, Dy))\| \leq \frac{1}{n}\|x - y\|. \quad (4.29)$$

Hence, $H((A, B), (C, D))$ is $\frac{29}{n}$ -mixed Lipschitz continuous with respect to A, B, C and D for $n = 9, 10$.

Let $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$f(x) = \left(\frac{1}{2}x_1 - \frac{4}{3}x_2, \frac{4}{3}x_1 + \frac{1}{2}x_2 \right), \quad g(x) = \left(\frac{1}{4}x_1 - \frac{3}{4}x_2, \frac{3}{4}x_1 + \frac{1}{4}x_2 \right), \quad \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

Suppose that $M : (\mathbb{R}^2 \times \mathbb{R}^2) \rightarrow \mathbb{R}^2$ is defined by

$$M(fx, gx) = fx - gx, \quad \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

Let for any $w \in \mathbb{R}^2$, we have

$$\begin{aligned} \langle M(fx, w) - M(fy, w), x - y \rangle &= \langle fx - w - fy + w, x - y \rangle \\ &= \langle fx - fy, x - y \rangle \\ &= \langle \left(\frac{1}{2}(x_1 - y_1) - \frac{4}{3}(x_2 - y_2) \right), \left(\frac{4}{3}(x_1 - y_1) + \frac{1}{2}(x_2 - y_2) \right), \\ &\quad (x_1 - y_1, x_2 - y_2) \rangle \\ &= \frac{1}{2} \{ (x_1 - y_1)^2 + 5(x_2 - y_2)^2 \} \\ &= \frac{1}{2} \|x - y\|^2. \end{aligned}$$

That is,

$$\langle u - v, x - y \rangle \geq \frac{1}{n} \|x - y\|^2, \quad \forall x, y \in X, u \in M(fx, w), v \in M(fy, w). \quad (4.30)$$

Hence, $M(f, g)$ is $\frac{1}{n}$ -strongly accretive with respect to f for $n = 2, 3$ and

$$\begin{aligned} \langle M(w, gx) - M(w, gy), x - y \rangle &= \langle w - gx - w + gy, x - y \rangle \\ &= -\langle gx - gy, x - y \rangle \\ &= -\langle \left(\frac{1}{4}(x_1 - y_1) - \frac{3}{4}(x_2 - y_2) \right), \left(\frac{3}{4}(x_1 - y_1) + \frac{1}{4}(x_2 - y_2) \right), \\ &\quad (x_1 - y_1, x_2 - y_2) \rangle \\ &= -\frac{1}{4} \{ (x_1 - y_1)^2 + (x_2 - y_2)^2 \} \\ &= -\frac{1}{4} \|x - y\|^2. \end{aligned}$$

That is,

$$\langle u - v, x - y \rangle \geq -\frac{1}{n} \|x - y\|^2, \quad \forall x, y \in X, u \in M(w, gx), v \in M(w, gy). \quad (4.31)$$

Hence, $M(f, g)$ is $\frac{1}{n}$ -relaxed accretive with respect to g for $n = 3, 4$.

From (4.31) and (4.32), $M(f, g)$ is symmetric accretive with respect to f and g . Also for any $x \in \mathbb{R}^2$, we have

$$\begin{aligned}
[H((A, B), (C, D)) + \rho M(f, g)](x) &= [H((Ax, Bx), (Cx, Dx)) + \rho M(fx, gx)] \\
&= (Ax + Bx + Cx + Dx) + \rho(fx - gx) \\
&= \left(\frac{1}{10}x_1, \frac{1}{10}x_2\right) + \left(-\frac{1}{5}x_1, -\frac{1}{5}x_2\right) + (2x_1, 2x_2) + (x_1, x_2) \\
&\quad + \rho \left(\left(\frac{1}{2}x_1 - \frac{4}{3}x_2, \frac{4}{3}x_1 + \frac{1}{2}x_2 \right) - \left(\frac{1}{4}x_1 - \frac{3}{4}x_2, \frac{3}{4}x_1 + \frac{1}{4}x_2 \right) \right) \\
&= \left(\frac{29}{10}x_1, \frac{29}{10}x_2 \right) + \rho \left(\frac{1}{4}x_1 - \frac{7}{12}x_2, \frac{7}{12}x_1 + \frac{1}{4}x_2 \right) \\
&= \left(\left(\frac{29}{10} + \frac{1}{4}\rho \right)x_1 - \frac{7}{12}\rho x_2, \left(\frac{29}{10} + \frac{7}{12}\rho \right)x_1 + \frac{1}{4}\rho x_2 \right),
\end{aligned}$$

it can be easily verify that the vector on right hand side generate the whole \mathbb{R}^2 . Therefore, we have

$$[H((A, B), (C, D)) + \rho M(f, g)]\mathbb{R}^2 = \mathbb{R}^2.$$

Hence, M is generalized $\alpha\beta$ - $H((., .), (., .))$ -mixed accretive with respect to (A, C) , (B, D) and (f, g) .

(iv) Let $F : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are defined by

$$F(x, y) = \frac{x}{4} + \frac{y}{5}, \quad \forall x, y \in \mathbb{R}^2.$$

Let $u \in \mathbb{R}^2$. Since S, T are Identity map, we have

$$\begin{aligned}
\langle F(x, u) - F(y, u), H((Ax, Bx), (Cx, Dx)) - H((Ay, By), (Cy, Dy)) \rangle &= \left\langle \frac{1}{4}x + \frac{1}{5}u - \left(\frac{1}{4}y + \frac{1}{5}u \right), \frac{29}{10}x - \frac{29}{10}y \right\rangle \\
&= \left\langle \frac{1}{4}(x_1, x_2) - \frac{1}{5}(y_1, y_2), \frac{29}{10}(x_1, x_2) - \frac{29}{10}(y_1, y_2) \right\rangle \\
&= \left\langle \left(\frac{1}{4}x_1 - \frac{1}{4}y_1, \frac{1}{5}x_2 - \frac{1}{5}y_2 \right), \left(\frac{29}{10}x_1 - \frac{29}{10}y_1, \frac{29}{10}x_2 - \frac{29}{10}y_2 \right) \right\rangle \\
&= \frac{29}{40} \{ (x_1 - y_1)^2 + (x_2 - y_2)^2 \} \\
&= \frac{29}{40} \|x - y\|^2,
\end{aligned}$$

that is,

$$\langle F(x, u) - F(y, u), H((Ax, Bx), (Cx, Dx)) - H((Ay, By), (Cy, Dy)) \rangle \geq \frac{29}{n} \|x - y\|^2, \quad (4.32)$$

similarly

$$\langle F(u, x) - F(u, y), H((Ax, Bx), (Cx, Dx)) - H((Ay, By), (Cy, Dy)) \rangle \geq \frac{29}{n} \|x - y\|^2. \quad (4.33)$$

From (4.32) and (4.33), we have F is $\frac{29}{n}$ -strongly accretive with respect to S and $H((A, B), (C, D))$ in the first argument for $n = 40, 41$ and $\frac{29}{n}$ -strongly accretive with respect to T and $H((A, B), (C, D))$ in the second argument for $n = 49, 50$.

(v) Let

$$\begin{aligned}
 \|F(x, u) - F(y, u)\|^2 &\leq \langle F(x, u) - F(y, u), F(x, u) - F(y, u) \rangle \\
 &= \langle \frac{1}{4}(x_1, x_2) - \frac{1}{4}(y_1, y_2), \frac{1}{4}(x_1, x_2) - \frac{1}{4}(y_1, y_2) \rangle \\
 &= \langle (\frac{1}{4}x_1 - \frac{1}{4}y_1, \frac{1}{4}x_2 - \frac{1}{4}y_2), (\frac{1}{4}x_1 - \frac{1}{4}y_1, \frac{1}{4}x_2 - \frac{1}{4}y_2) \rangle \\
 &= \frac{1}{4}\{(x_1 - y_1)^2 + (x_2 - y_2)^2\} \\
 &= \frac{1}{16}\|x - y\|^2,
 \end{aligned}$$

that is

$$\|F(x, u) - F(y, u)\| \leq \frac{1}{n}\|x - y\|, \quad (4.34)$$

similarly

$$\|F(u, x) - F(u, y)\| \leq \frac{1}{n}\|x - y\|. \quad (4.35)$$

From (4.33) and (4.34), F is $\frac{1}{n}$ -Lipschitz continuous in the first argument for $n = 3, 4$ and $\frac{1}{n}$ -Lipschitz continuous in the second argument for $n = 4, 5$.

(vi) For the constants

$$\begin{aligned}
 l_1 = l_2 = 1, \mu_1 = 10, \gamma_1 = 2, \mu_2 = 5, \gamma_2 = 1, \alpha_1 = 0.1, \beta_1 = 0.2, \\
 \alpha = 0.5, \beta = 0.25, \sigma = 0.725, \delta = 0.580, \epsilon_1 = 0.25, \epsilon_2 = 0.2, \tau = 2.9, \\
 q = 2, r = 2.9, m = 0.25.
 \end{aligned}$$

obtained from above conditions (i) to (vi), one can easily verify, for $c_q = 1$ and $\rho = 3.8$ and , that the condition (vi) of Theorem 4.6, given by

$$0 < \sqrt[q]{\tau^q + c_q \rho^q (\epsilon_1 l_1 + \epsilon_2 l_2)^q} - \rho q (\sigma + \delta) \tau^q < r + \rho m,$$

where $r = (\mu_1 \alpha_1^q - \mu_2 \beta_1^q) + (\gamma_1 + \gamma_2)$ and $m = \alpha - \beta$, and $\alpha > \beta, \mu_1 > \mu_2, \alpha_1 > \beta_1$ and $\gamma_1, \gamma_2, \rho > 0$ holds.

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